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Second order nonlinear multivalued boundary problems in Hilbert spaces

Honglv Ma *, Xingmei Xue

Department of Mathematics, Southeast University, Nanjing 210018, PR China

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Abstract

In this paper we consider second order differential inclusions in real Hilbert space, namely $p(t) \cdot x''(t) + r(t) \cdot x'(t) \in Ax(t) + F(t, x(t))$, a.e. on $[0, T]$, under the nonlinear boundary conditions. Using techniques from multivalued analysis and the theory of operators of monotone type, we prove the existence of solutions for both the 'convex' and 'nonconvex' problems. Finally, we present a special case of interest, which fit into our framework, illustrating the generality of our results.

Keywords: Maximal monotone operator; Compact sets; Lower semicontinuous; Weakly upper semicontinuous; Second order differential inclusions; Contractible values

1. Introduction

In this paper, we consider the following second order multivalued boundary value problem:

$$\begin{cases} p(t) \cdot x''(t) + r(t) \cdot x'(t) \in Ax(t) + F(t, x(t)), & \text{a.e. } t \in I, \\ (x'(0), -\tilde{r}(T)x'(T)) \in \xi(x(0) - a, x(T) - b), \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail address: mhl226@163.com (H. Ma).

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here $F : I \times H \to 2^H \setminus \emptyset$ is a multifunction, A is a maximal monotone operator in a real Hilbert space H, ξ is maximal monotone in $H \times H$; $a, b \in D(A)$; $p, r \in C(I; R)$; $\tilde{r}(t) = \exp(\int_0^t r(s)/p(s)\,ds)$, $I = [0, T]$.

Second order differential inclusions with various boundary conditions have been studied recently by Benchohra and Ntouyas [4], Gasiński and Papageorgiou [7], Halidias and Papageorgiou [9,10], Kravvaritis and Papageorgiou [11], Kyritsi et al. [12] and Ma and Xue [13]. In [7] and [12], the authors study nonlinear periodic boundary value problem in R^N with a maximal monotone term, whose approach is based on results from multivalued analysis and the theory of monotone operators. Benchohra and Ntouyas [4] used a fixed point theorem for condensing maps, coupled with the method of upper and lower solutions to study semilinear periodic problem in R. In [9,10], the boundary conditions are nonlinear and include the classical ones (Dirichlet, Neumann and periodic). However, in all these works but Ma and Xue [13], the authors limit the essential space to finite dimensional space R^N $(1 \leqslant N < \infty)$. In Ma and Xue [13], the authors obtained a existence theorem for differential inclusions in separable Hilbert space and limit their study to the "convex" problem (i.e., F is assumed to be convex valued).

In this paper, we extend the essential space to a general Hilbert space H when we assume the operator F is convex valued and examine both the 'convex' and 'nonconvex' problems. The price that we pay to achieve this generalization, is that we have to strengthen the compactness hypothesis on $F(t, x)$. But such hypothesis is obviously true in [4,7,9–12]. At the same time, our formulation of the problem (1.1) is general. For example, the boundary conditions in [9] are situations $a = b = 0$ in our work; the inclusion $y'' \in F(t, y)$ in [4] is situation $p(t) \equiv 1$, $r(t) \equiv 0$, $A \equiv 0$ in problem (1.1). Certainly, since in [12] the time differential operator is much more general (it concludes the vector ordinary p-Laplacian), so in that respect [12] is more general. Finally, we also mention that our work can be viewed as an extension of Apreutesei [2]. In [2], the author study the problem of the form (1.1) with $F(t, x) \equiv f(t)$.

Our approach will be based on notions and results from multivalued analysis and the theory of nonlinear operators of monotone type.

2. Preliminaries

Let X be a Banach space, (Ω, Σ) be a measurable space, we introduce the following notation:

$$P_{f(c)}(X) := \{K \subseteq X\colon\ K \text{ is nonempty, closed (and convex)}\},$$
$$P_{k(c)}(X) := \{K \subseteq X\colon\ K \text{ is nonempty, compact (and convex)}\}.$$

A multifunction $F : \Omega \to P_f(X)$ is said to be graph measurable, if $\operatorname{Gr} F := \{(\omega, x) \in \Omega \times X\colon\ x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$, where $\mathcal{B}(X)$ being the Borel σ-field of X.

Let Y, Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z \setminus \emptyset$ is said to be lower semicontinuous (lsc) (respectively upper semicontinuous (usc)), if for every closed set $C \subseteq Z$, the set $G^+(C) := \{y \in Y\colon\ G(y) \subseteq C\}$ (respectively $G^-(C) := \{y \in Y\colon\ G(y) \cap C \neq \emptyset\}$) is closed in Y. We say that G is weakly usc if $G^-(C)$ is closed for all weakly closed $C \subseteq Z$; and G is ϵ–δ-usc if for every $x_0 \in Y$ and $\epsilon > 0$ there is $\delta = \delta(x_0, \epsilon) > 0$ such that $G(x) \subset G(x_0) + B_\epsilon(0)$ for all $x \in B_\delta(x_0)$. Evidently usc is stronger than weakly usc. If Y, Z are both metric spaces, then the above definition of lsc is equivalent to saying that for all $z \in Z$, $y \mapsto d_Z(z, G(y)) := \inf\{d_Z(z, v)\colon v \in G(y)\}$ is usc as an R_+-valued function.

Let H be a real Hilbert space, with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The inner product and the norm of $H \times H$ are also denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. A map $A : D(A) \subset H \to 2^H \setminus \emptyset$ is said to be monotone, if for all $(x, x^*), (y, y^*) \in \operatorname{Gr} A$, we have $\langle x - y, x^* - y^*\rangle \geqslant 0$. If additionally, the fact that $\langle x - y, x^* - y^*\rangle \geqslant 0$ for all $(x, x^*) \in \operatorname{Gr} A$, implies that $(y, y^*) \in \operatorname{Gr} A$, then we say that A is maximal monotone. It is easy to see, that a maximal monotone map A has a demiclosed graph, i.e., $\operatorname{Gr} A$ is sequentially closed in $H \times H_w$ (here by H_w, we denote the space H furnished with the weak topology). Moreover, if A is maximal monotone, then we can define the single-valued operator $J_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$ (the Yosida approximation of A) on H for $\lambda > 0$.

Denote

$$\begin{aligned} D(\mathcal{A}) = \big\{&x \in L^2(I; H)\colon \exists y \in L^2(I; H),\ \text{s.t. } y(t) \in A\big(x(t)\big),\ x(t) \in D(A), \\ &\text{a.e. } t \in I\big\}, \\ \mathcal{A}(x) = \big\{&y \in L^2(I; H)\colon\ y(t) \in A\big(x(t)\big),\ \text{a.e. } t \in I\big\}, \end{aligned}$$

then $\mathcal{A}$ is said to be the realization of A in $L^2(I; H)$. We know that, if A is maximal monotone in H, then $\mathcal{A}$ is also maximal monotone in $L^2(I; H)$ and, if $A_\lambda, \mathcal{A}_\lambda$ are Yosida approximations of A, respectively $\mathcal{A}$, then

$$(\mathcal{A}_\lambda x)(t) = A_\lambda\big(x(t)\big), \quad \forall t \in I,\ \forall x \in D(\mathcal{A}).$$

For the sake of simplicity, we denote the point $y \in A(x)$ by $A(x)$ in the following. For more details on monotone maps see the books of Barbu [3].

Let $T > 0$ and $I = [0, T]$. We define the function $\tilde{r}(t) = \exp(\int_0^t r(s)/p(s)\,ds)$ on I, where $p, r : I \to R$ are two continuous functions with $p(t) \geqslant c > 0$, $\forall t \in I$. We introduce the space $E = L^2_{\tilde{r}/p}(I; H)$, which means $(L^2(I; H), \|\cdot\|_2)$ with the weighted function $\tilde{r}/p$. Therefore, the inner product and the norm of E are defined by

$$\langle\!\langle x, y\rangle\!\rangle = \int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle x(t), y(t)\big\rangle\,dt, \quad \forall x, y \in E,$$

$$|x|^2 = \int_0^T \frac{\tilde{r}(t)}{p(t)}\big\|x(t)\big\|^2\,dt, \quad \forall x \in E.$$

Denote by " $\to$" (respectively " $\rightharpoonup$") the strong (weak) convergence in all the infinite dimensional spaces involved.

Remark. The spaces $L^2(I; H)$ and E contain the same functions and have equivalent norms.

Definition 2.1. Let $A : D(A) \subset H \to 2^H \setminus \emptyset$, $\xi : D(\xi) \subset H \times H \to 2^{H\times H} \setminus \emptyset$ be two maximal monotone maps. The operator A is said to be ξ-monotone if

$$\begin{aligned}&\big\langle (A_\lambda x_1 - A_\lambda x_2, A_\lambda y_1 - A_\lambda y_2), (z_1, z_2)\big\rangle\\ &\quad= \langle A_\lambda x_1 - A_\lambda x_2, z_1\rangle + \langle A_\lambda y_1 - A_\lambda y_2, z_2\rangle \geqslant 0\end{aligned}$$

holds for every $(z_1, z_2) \in \xi(x_1 - x_2, y_1 - y_2)$, $(x_1 - x_2, y_1 - y_2) \in D(\xi)$ and $\lambda > 0$, here A_λ is Yosida approximation of A.

Definition 2.2. Let X be a Banach space and $B \subset X$. Then the set B is said to be contractible, if there exists a continuous mapping $h : [0, 1] \times B \to B$ and $x_0 \in X$, such that $h(0, x) = x_0$, $h(1, x) = x$, $\forall x \in B$.

Now, let us record some facts about set valued maps which will be useful later on.

Lemma 2.1 [6, Theorem 3]. *Let X be a separable metric space, Y be a Banach space, $N : X \to 2^{L^1(I;Y)} \setminus \emptyset$ be lsc with closed decomposable values. Then N admits a continuous selector, i.e., exists a continuous operator $G : X \to L^1(I; Y)$, such that $G(x) \in N(x)$, $\forall x \in X$.*

Lemma 2.2 [5, Proposition 2]. *Let X, Y be Banach spaces, Ω be a nonempty subset of X and $F : \Omega \to 2^Y \setminus \emptyset$ have weakly compact convex values. Then F is weakly usc iff $\{x_n\} \subset \Omega$ with $x_n \to x_0 \in \Omega$ and $y_n \in F(x_n)$ implies $y_{n_k} \rightharpoonup y_0 \in F(x_0)$ for some subsequence $\{y_{n_k}\}$ of $\{y_n\}$.*

Lemma 2.3 [5, Lemma 1]. *Let X be a Banach space, $\emptyset \neq D \subset X$ compact convex and $F : D \to 2^D \setminus \emptyset$ be usc with closed contractible values. Then F has a fixed point.*

In the proof of our main results, the following lemma will play a crucial role.

Let X be a real Banach space, with dual $(X^*, \|\cdot\|_*)$. The duality map in X is defined by $J(x) = \{f \in X^*\colon f(x) = \|x\|^2 = \|f\|_*^2\}$, $\forall x \in X$.

Denote by $C_w = C(I; X^*; \sigma(X^*, X))$ the space of bounded continuous functions $x : I \to X^*$, when X^* is equipped with its weak-star topology of $\sigma(X^*, X)$.

Lemma 2.4 [8, Lemma 2.6]. *Let X be a uniformly smooth Banach space. If $K \subset C(I; X)$ is a bounded, equicontinuous set of functions, then*

(a) *$J(K) = \{t \to Jg(t)\colon g \in K\}$ is relatively compact in C_w;*
(b) *If $\{g_n\} \subset J(K)$, and $g_n \to g$ in C_w, then $g \in (L^1(I; X))^*$ and $\sup\{|(g_n(t) - g(t))(x)|\colon x \in K\} \to 0$ $(n \to \infty)$, for any compact $K \subset X$ and $t \in I$.*

3. Main results

In this section, we are looking for solutions of (1.1) in the Sobolev space $W^{2,2}(I; H)$. We shall prove existence theorems for both the convex and nonconvex problems. We consider a convex existence theorem at first. In this case, our hypotheses on the multifunction $F(t,x)$ are the following.

$H(F)_1$ $F: I \times H \to P_{kc}(H)$ is a multifunction such that
- (a) for $\forall x \in H$, $t \to F(t,x)$ admits a measurable selector;
- (b) for a.e. $t \in I$, $x \to F(t,x)$ is weakly usc;
- (c) for a.e. $t \in I$, $|F(t,x)| = \sup\{\|v\|\colon v \in F(t,x)\} \leqslant \phi(t) + \psi(t) \cdot \|x\|^\alpha$, here $0 \leqslant \alpha < 1$; $\phi, \psi \in L^2(I; R_+)$;
- (d) for $\forall K > 0$, a.e. $t \in I$, the set $B_K(t) = \{v \in F(t,x)\colon \|x\| \leqslant K\}$ is relatively compact.

Remark. Here we impose a compactness condition on F, i.e., $H(F)_1$(d). However, if H is a finite dimensional space, then $H(F)_1$(c) implies $H(F)_1$(d), which shows that $H(F)_1$(d) is also true in [4,7,9–12].

The hypotheses on the maps $A(\cdot)$ and $\xi(\cdot,\cdot)$ are the following.

$H(A)_1$ $A: D(A) \subset H \to 2^H \setminus \emptyset$ is maximal monotone in H, A is ξ-monotone and $0 \in D(A)$.

$H(\xi)_1$ $\xi: D(\xi) \subset H \times H \to 2^{H\times H}$ is a maximal monotone map with $(0,0) \in \xi(0,0)$ and, exists $\omega > 0$, $p > \max\{\frac{2}{2-\alpha}, \alpha + 1\}$, such that

$$\left\langle (z_1, z_1') - (z_2, z_2'), (x_1, y_1) - (x_2, y_2)\right\rangle \geqslant \omega \left| \left\|(x_1, y_1)\right\| - \left\|(x_2, y_2)\right\| \right|^p$$

holds for every $(x_i, y_i) \in D(\xi)$, $(z_i, z_i') \in \xi(x_i, y_i)$ $(i = 1, 2)$.

Remark. Without loss of generality, we may assume $0 \in A(0)$. Otherwise we would consider $\tilde{A}(x) = A(x) - A^0(0)$ and $\tilde{F}(t,x) = F(t,x) + A^0(0)$, respectively.

Theorem 3.1. *Let H be a real Hilbert space and $p, r: I \to R$ be continuous functions with $p(t) \geqslant c > 0$, $\forall t \in I$. If $H(A)_1$, $H(F)_1$ and $H(\xi)_1$ hold*; $a, b \in D(A)$; *then problem* (1.1) *has a solution $x \in W^{2,2}(I; H)$.*

The proof of Theorem 3.1 is based on the following lemmas.

Lemma 3.1. *Let assumptions of Theorem* 3.1 *be satisfied. Then for every $f \in L^2(I; H)$, problem*

$$\begin{cases} -(p(t)\cdot x''(t) + r(t)\cdot x'(t)) + Ax(t) \ni f(t), & a.e.\ t \in I, \\ (x'(0), -\tilde{r}(T)x'(T)) \in \xi(x(0) - a, x(T) - b) \end{cases} \tag{3.1}$$

has a unique solution $x \in W^{2,2}(I; H)$.

Proof. Let $B: E \to E$ be defined by $B(x) = -(px'' + rx') = -\frac{p}{\tilde{r}}(\tilde{r}x')'$, $\forall x \in D(B)$, where

$$D(B) = \left\{x \in W^{2,2}(0,T; H)\colon \left(x'(0), -\tilde{r}(T)x'(T)\right) \in \xi\left(x(0) - a, x(T) - b\right)\right\}.$$

Then $B : E \to E$ is maximal monotone (see, e.g., [2, Proposition 2.2]). Denote by $\mathcal{A}$ the realization of A in E, then $B + \mathcal{A}_\lambda$ is maximal monotone in E (see, e.g. [3, Chapter 2, Corollary 1.3]). Then, for every $f \in L^2(I; H)$ and $\lambda > 0$, there is a unique $x_\lambda \in D(B)$ satisfying

$$\begin{cases} -(p(t) \cdot x_\lambda''(t) + r(t) \cdot x_\lambda'(t)) + \mathcal{A}_\lambda x_\lambda(t) + \lambda x_\lambda(t) \ni f(t), \quad \text{a.e. } t \in I, \\ (x_\lambda'(0), -\tilde{r}(T)x_\lambda'(T)) \in \xi(x_\lambda(0) - a, x_\lambda(T) - b). \end{cases} \tag{3.2}$$

Therefore, as in the proof of Theorem 2.3 of [1], we can prove that $\|x_\lambda(0)\| \leqslant k_1$, $\|x_\lambda(T)\| \leqslant k_2$, $\forall \lambda > 0$, where $k_1, k_2 > 0$ are independent of λ.

Next, we need to show that $\|x_\lambda\|_\infty$, $|x_\lambda'|$, $|x_\lambda''|$, $|\mathcal{A}_\lambda x_\lambda|$ are bounded. Indeed, let y be the unique solution of $y'' \in \mathcal{A}y$, $y(0) = a$, $y(T) = b$, then we have

$$\big\|A_\lambda\big(y(t)\big)\big\| \leqslant \big\|A^0\big(y(t)\big)\big\| \leqslant \big\|y''(t)\big\|.$$

Multiplying $Bx_\lambda + \mathcal{A}_\lambda x_\lambda + \lambda x_\lambda - \mathcal{A}_\lambda y \ni f(t) - \mathcal{A}_\lambda y$ by $x_\lambda - y$ in E, and integrating by parts, we get

$$\begin{aligned} &\int_0^T \tilde{r}(t)\big\langle x_\lambda'(t), x_\lambda'(t) - y'(t)\big\rangle\, dt \\ &\quad \leqslant \int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle \lambda x_\lambda(t), y(t)\big\rangle\, dt + \int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle f(t) - A_\lambda\big(y(t)\big), x_\lambda(t) - y(t)\big\rangle\, dt. \end{aligned}$$

Using Hölder inequality and Cauchy inequality, we have that

$$c|x_\lambda'|^2 \leqslant k_3|x_\lambda'| + k_4|x_\lambda| + k_5, \tag{3.3}$$

here $k_3, k_4, k_5 > 0$ are independent of λ (without loss of generality, we may assume $0 < \lambda \leqslant 1$). Then, by (3.3) and the inequality $\|x_\lambda(t)\| \leqslant \|x_\lambda(0)\| + \int_0^T \|x_\lambda'(t)\|\, dt$, we have that $|x_\lambda'| \leqslant k_6$, $\|x_\lambda\|_\infty \leqslant k_7$, $\forall \lambda \in (0, 1]$. Multiplying (3.2) by $\mathcal{A}_\lambda(x_\lambda)$ in E, we get

$$-\int_0^T \big\langle\big(\tilde{r}(t)x_\lambda'(t)\big)', \mathcal{A}_\lambda x_\lambda(t)\big\rangle\, dt + |\mathcal{A}_\lambda x_\lambda|^2 \leqslant |f| \cdot |\mathcal{A}_\lambda x_\lambda|. \tag{3.4}$$

Observing that $\langle x_\lambda'(t), (\mathcal{A}_\lambda x_\lambda)'(t)\rangle \geqslant 0$, a.e. $t \in I$ and A is ξ-monotone, we have that

$$-\int_0^T \big\langle\big(\tilde{r}(t)x_\lambda'(t)\big)', \mathcal{A}_\lambda x_\lambda(t)\big\rangle\, dt \geqslant -\tilde{r}(T)\big\langle x_\lambda'(T), A_\lambda b\big\rangle + \big\langle x_\lambda'(0), A_\lambda a\big\rangle. \tag{3.5}$$

Let $\eta_\lambda(t) = (1 - \frac{t}{T})A_\lambda a + \frac{t}{T}A_\lambda b$, then we get

$$\langle\!\langle Bx_\lambda, \eta_\lambda\rangle\!\rangle = -\tilde{r}(T)\big\langle x_\lambda'(T), A_\lambda b\big\rangle + \big\langle x_\lambda'(0), A_\lambda a\big\rangle + \int_0^T \tilde{r}(t)\big\langle x_\lambda'(t), \eta_\lambda'(t)\big\rangle\, dt.$$

This equality in conjunction with (3.4) and (3.5), yields

$$|\mathcal{A}_\lambda x_\lambda|^2 \leqslant |f| \cdot |\mathcal{A}_\lambda x_\lambda| + |Bx_\lambda| \cdot |\eta_\lambda| + P|\eta_\lambda'| \cdot |x_\lambda'|,$$

here $P = \sup\{|p(t)|:\ t \in I\}$. Then, by $|Bx_\lambda| \leqslant \lambda|x_\lambda| + |\mathcal{A}_\lambda x_\lambda| + |f|$ and the definition of η_λ, we can easily prove that $|\mathcal{A}_\lambda x_\lambda|$ is bounded. According to (3.2), we also have that $|x''_\lambda|$ is bounded.

Now, we can complete the proof of the lemma. Given $\lambda, \eta > 0$, applying (3.2) again, we have that $Bx_\lambda - Bx_\eta + \mathcal{A}_\lambda x_\lambda - \mathcal{A}_\eta x_\eta + \lambda x_\lambda - \eta x_\eta = 0$. Taking inner product with $x_\lambda - x_\eta$ in E, and using the estimate above, we obtain that

$$c|x'_\lambda - x'_\eta|^2 \leqslant (\lambda + \eta)k_8 \to 0 \quad (\lambda, \eta \to 0_+),$$

here $k_8 > 0$ is independent of λ and η. Then, without loss of generality, we may assume $x'_\lambda \to v$ in E as $\lambda \to 0_+$. This and the boundedness of $\{x''_\lambda\}$ in E imply the convergence of $\{x'_\lambda\}$ in $C(I; H)$ and the weak convergence of $\{x''_\lambda\}$ in E, i.e., $x''_\lambda \rightharpoonup v'$ in E as $\lambda \to 0_+$.

Because $\|x_\lambda(0)\|$ is bounded and so, by passing to a subsequence if necessary, we may assume that $x_\lambda(0) \rightharpoonup l$. Also, since $x_\lambda(t) = x_\lambda(0) + \int_0^t x'_\lambda(s)\,ds$, we obtain that $x_\lambda(t) \rightharpoonup l + \int_0^t v(s)\,ds$. Set $x(t) = l + \int_0^t v(s)\,ds$, then $x'(t) = v(t)$, $\forall t \in I$. Thus we have $x_\lambda(t) \rightharpoonup x(t)$ in H and $Bx_\lambda \rightharpoonup Bx$ in E as $\lambda \to 0_+$. These and the convergence of $\{x'_\lambda\}$ in $C(I; H)$ yield that $x_\lambda \rightharpoonup x$, $\mathcal{A}_\lambda x_\lambda = -Bx_\lambda - \lambda x_\lambda + f \rightharpoonup -Bx + f$ in E, and $\langle\!\langle Bx_\lambda, x_\lambda\rangle\!\rangle \to \langle\!\langle Bx, x\rangle\!\rangle$ as $\lambda \to 0_+$. Then, we can conclude that $x \in D(\mathcal{A})$ and $-Bx + f \in \mathcal{A}x$ since $\mathcal{A}$ is maximal monotone in E (see, e.g., [3, Chapter 2, Proposition 1.1]). Similarly, because of ξ maximal monotone, we get that ξ is closed in $H \times H_w$, so $(x'(0), -\tilde{r}(T)x'(T)) \in \xi(x(0) - a, x(T) - b)$, which shows that x is a solution of problem (3.1).

Now we need to prove the uniqueness of solution. Let x, y be two solutions of problem (3.1), then we have $Bx - By + \mathcal{A}x - \mathcal{A}y \ni 0$. Taking inner product with $x - y$ in E, it follows that $c|x' - y'|^2 \leqslant 0$. Since $x', y' \in C(I; H)$, we have that $x'(t) = y'(t)$, $x(t) = y(t) + d$, $\forall t \in I$, here d is a constant. Thus we get

$$\bigl(x'(0), -\tilde{r}(T)x'(T)\bigr) \in \xi\bigl(x(0) - a, x(T) - b\bigr) \cap \xi\bigl(x(0) - a - d, x(T) - b - d\bigr).$$

As ξ is one-to-one, we obtain that $d = 0$, i.e., $x(t) = y(t)$, $\forall t \in I$. Uniqueness is proved. □

Now, we can define $S : L^2(I; H) \to C(I; H)$ by $S(f) = x$, where x is the unique solution of problem (3.1).

Lemma 3.2. *Let assumptions of Theorem* 3.1 *be satisfied, K be a bounded subset of $L^2(I; H)$. Then $S(K) \subset C(I; H)$ is a bounded equicontinuous family of functions, and $S(K) \subset W^{2,2}(I; H)$ is bounded.*

Proof. Assume $\|f\|_2 \leqslant M$, $\forall f \in K$. Given $f \in K$, let $x = S(f)$ and $\eta(t) = \frac{t}{T}b + (1 - \frac{t}{T})a$, $\forall t \in I$. Then $|\eta|$, $\|A^0(b)\|$ and $\|A^0(a)\|$ are independent of f. For the sake of simplicity, we may assume

$$\|a\| + \|b\| + |\eta| + \bigl\|A^0(b)\bigr\| + \bigl\|A^0(a)\bigr\| \leqslant C_1, \quad R_1 \leqslant \tilde{r}(t) \leqslant R_2.$$

Multiplying (3.1) with $x - \eta$ and integrating by parts, we have that

$$-\tilde{r}(T)\bigl\langle x'(T), x(T) - b\bigr\rangle + \bigl\langle x'(0), x(0) - a\bigr\rangle$$

$$
\begin{aligned}
&+\int_0^T \tilde r(t)\Big\langle x'(t), x'(t)-\frac{1}{T}(b-a)\Big\rangle\,dt\\
&+\int_0^T \frac{\tilde r(t)}{p(t)}\Big\langle A\big(x(t)\big)-A^0(b), \frac{t}{T}\big(x(t)-b\big)\Big\rangle\,dt\\
&+\int_0^T \frac{\tilde r(t)}{p(t)}\Big\langle A^0(b), \frac{t}{T}\big(x(t)-b\big)\Big\rangle\,dt\\
&+\int_0^T \frac{\tilde r(t)}{p(t)}\Big\langle A\big(x(t)\big)-A^0(a), \Big(1-\frac{t}{T}\Big)\big(x(t)-a\big)\Big\rangle\,dt\\
&+\int_0^T \frac{\tilde r(t)}{p(t)}\Big\langle A^0(a), \Big(1-\frac{t}{T}\Big)\big(x(t)-a\big)\Big\rangle\,dt\\
&=\int_0^T \frac{\tilde r(t)}{p(t)}\big\langle f(t), x(t)-\eta(t)\big\rangle\,dt.
\end{aligned}
$$

From $H(\xi)_1$, we have that

$$
\begin{aligned}
&\omega\big\|x(0)-a\big\|^p+R_1\|x'\|_2^2-\frac{1}{T}\int_0^T \tilde r(t)\big\langle x'(t), b-a\big\rangle\,dt\\
&\leqslant C_1\int_0^T \frac{\tilde r(t)}{p(t)}\big(\big\|x(t)-b\big\|+\big\|x(t)-a\big\|\big)\,dt\\
&\quad+\int_0^T \frac{\tilde r(t)}{p(t)}\big\langle f(t), x(t)-\eta(t)\big\rangle\,dt. \qquad (3.6)
\end{aligned}
$$

Note also that

$$
\frac{1}{T}\int_0^T \tilde r(t)\big\langle x'(t), b-a\big\rangle\,dt\leqslant\frac{C_1R_2}{T}\int_0^T \big\|x'(t)\big\|\,dt\leqslant \frac{R_1}{2}\|x'\|_2^2+C_3,
$$

and

$$
C_1\int_0^T \frac{\tilde r(t)}{p(t)}\big(\big\|x(t)-b\big\|+\big\|x(t)-a\big\|\big)\,dt
$$

$$\leqslant C_1 \cdot \int_0^T \frac{\tilde r(t)}{p(t)}\bigl(2\bigl\|x(t)\bigr\| + C_1\bigr)\,dt \leqslant 2C_1 \cdot C_2 \cdot \|x\|_\infty + C_1^2 \cdot C_2, \tag{3.7}$$

and

$$\int_0^T \frac{\tilde r(t)}{p(t)}\bigl\langle f(t), x(t) - \eta(t)\bigr\rangle\,dt \leqslant \sqrt{C_2}\,|f| \cdot \|x\|_\infty + C_1|f|,$$

we have

$$\bigl\|x(0)\bigr\|^p + \|x'\|_2^2 \leqslant C_4 \cdot \bigl(\|x\|_\infty + \|x\|_\infty \cdot \|f\|_2 + \|f\|_2\bigr) + C_5, \tag{3.8}$$

where $C_2 = \int_0^T \frac{\tilde r(t)}{p(t)}\,dt$, and C_3, C_4, C_5 are independent of f. Apply

$$\bigl\|x(t)\bigr\| \leqslant \bigl\|x(0)\bigr\| + \int_0^t \bigl\|x'(s)\bigr\|\,ds \leqslant \bigl\|x(0)\bigr\| + \sqrt{T}\,\|x'\|_2, \tag{3.9}$$

$\|f\|_2 \leqslant M$ and (3.8) again, we have that $\|x(0)\| + \|x'\|_2 \leqslant C_6$, here C_6 is independent of f. By (3.9), we know that $S(K)$ is bounded in $C(I;H)$. Therefore, using $\|x(t) - x(s)\| \leqslant \int_s^t \|x'(\tau)\|\,d\tau \leqslant \sqrt{t-s} \cdot C_6$, $\forall x \in S(K)$, $0 \leqslant s \leqslant t \leqslant T$, we can obtain that $S(K)$ is a bounded equicontinuous set in $C(I;H)$.

To prove the boundedness of $S(K)$ in $W^{2,2}(I;H)$, we only need to show that $\{x''\colon x \in S(K)\}$ is bounded in $L^2(I;H)$. For this purpose, we assume that x_λ is the solution of problem (3.2) with $f \in K$. Then, as in the proof of Lemma 3.1, we have that

$$|\mathcal{A}_\lambda x_\lambda|^2 \leqslant |f| \cdot |\mathcal{A}_\lambda x_\lambda| + |Bx_\lambda| \cdot |\eta_\lambda| + P|\eta_\lambda'| \cdot |x_\lambda'|,$$

hence $|\mathcal{A}_\lambda x_\lambda|^2 \leqslant k_8(|f|^2 + \lambda|x_\lambda|^2 + |x_\lambda'| + 1)$ for all $\lambda > 0$ and k_8 is independent of λ. Note also that $\|x_\lambda''\|_2 \leqslant k_9(|\mathcal{A}_\lambda x_\lambda| + \lambda|x_\lambda| + |x_\lambda'| + |f|)$, $x_\lambda'' \rightharpoonup x''$ in E, $x_\lambda \to x$ in $C^1(I;H)$, and K is bounded in $L^2(I;H)$, $S(K)$ is bounded in $W^{1,2}(I;H)$, let $\lambda \to 0+$, we have that $\{\|x''\|_2\colon x \in S(K)\}$ is bounded, which implies that $S(K)$ is bounded in $W^{2,2}(I;H)$. □

Lemma 3.3. *Let assumptions of Theorem* 3.1 *be satisfied. Denote* $K = \{f \in L^2(I;H)\colon \|f(t)\| \leqslant y(t),\ f(t) \in k(t),\ \text{a.e. } t \in I\}$, *where* $k(t)$ *is relatively compact for almost every* $t \in I$, $y \in L^2(I;R_+)$. *Then* $S(K)$ *is relatively compact in* $C(I;H)$. *Moreover, if* $\{f_n\} \subset K$ *and* $f_n \rightharpoonup f_0$ *in* $L^2(I;H)$, *then* $S(f_n) \to S(f_0)$ *in* $C^1(I;H)$ *as* $n \to \infty$.

Proof. Denote $\bar K = \{f \in L^2(I;H)\colon \|f(t)\| \leqslant y(t),\ f(t) \in \overline{\mathrm{conv}}\,k(t),\ \text{a.e. } t \in I\}$. Because $\overline{\mathrm{conv}}\,k(t)$ is compact and convex in H, so $\bar K$ is closed and convex in $L^2(I;H)$ and $K \subseteq \bar K$. Therefore, to prove the relative compactness of $S(K)$, it is suffices to show that $S(\bar K)$ is relatively compact in $L^2(I;H)$. Let $\{x_n\}_{n\geqslant 1}$ be any series in $S(\bar K)$, then there exists $\{f_n\} \subset \bar K$, such that

$$\begin{cases} -(p(t) \cdot x_n''(t) + r(t) \cdot x_n'(t)) + \mathcal{A}x_n(t) \ni f_n(t), & \text{a.e. } t \in I, \\ (x_n'(0), -\tilde r(T)x_n'(T)) \in \xi(x_n(0) - a, x_n(T) - b). \end{cases} \tag{3.10}$$

As $\{f_n\} \subset \bar K$, by passing to a subsequence if necessary, we may assume $f_n \rightharpoonup f_0$ in $L^2(I;H)$. Because $\bar K$ is closed and convex in $L^2(I;H)$, we know that $\bar K$ is weakly closed in $L^2(I;H)$, then $f_0 \in \bar{K}$. Moreover, $S(\bar{K})$ is a bounded equicontinuous set in $C(I;H)$ by Lemma 3.2. Therefore, according to Lemma 2.4, we know that $\{x_n\}_{n\geqslant 1}$ has a convergence subsequence in C_w. Without loss of generality, we may assume $x_n \to x$ in C_w. Applying Lemma 2.4 again, we obtain $x \in (L^1(I;H))^* \subseteq L^2(I;H)$. Let $x_0 = S(f_0)$, then

$$-(px_0'' + rx_0') + \mathcal{A}x_0 \ni f_0. \tag{3.11}$$

From (3.10)–(3.11), we have

$$Bx_n - Bx_0 + \mathcal{A}x_n - \mathcal{A}x_0 \ni f_n - f_0.$$

Taking inner product with $x_n - x_0$ in E and integrating by parts, we have that

$$\begin{aligned}0 &\leqslant -\tilde{r}(T)\big\langle x_n'(T) - x_0'(T), x_n(T) - x_0(T)\big\rangle + \big\langle x_n'(0) - x_0'(0), x_n(0) - x_0(0)\big\rangle \\ &\quad + \int_0^T \tilde{r}(t)\big\|x_n'(t) - x_0'(t)\big\|^2\,dt \\ &\leqslant \int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle f_n(t) - f_0(t), x_n(t) - x(t)\big\rangle\,dt \\ &\quad + \int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle f_n(t) - f_0(t), x(t) - x_0(t)\big\rangle\,dt.\end{aligned} \tag{3.12}$$

The last integral tends to zero as $n \to \infty$, since $f_n \rightharpoonup f_0$ in $L^2(I;H)$ and $x - x_0 \in L^2(I;H)$. For the first one, because $f_n(t), f_0(t) \in \overline{\mathrm{conv}}\,k(t)$, a.e. $t \in I$, $x_n \to x$ in C_w, we can make use of Lemma 2.4 and get

$$\big|\big\langle f_n(t), x_n(t) - x(t)\big\rangle\big| \leqslant \sup\big\{\big|\big\langle v, x_n(t) - x(t)\big\rangle\big|\colon v \in \overline{\mathrm{conv}}\,k(t)\big\} \to 0 \quad (n \to \infty).$$

On the other hand, we have

$$\big|\big\langle f_n(t), x_n(t) - x(t)\big\rangle\big| \leqslant y(t)\big(\|x_n\|_\infty + \|x\|_\infty\big).$$

By virtue of Lebesgue's control convergence theorem, we have that

$$\int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle f_n(t), x_n(t) - x(t)\big\rangle\,dt \to 0 \quad (n \to \infty)$$

and

$$\int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle f_0(t), x_n(t) - x(t)\big\rangle\,dt \to 0 \quad (n \to \infty).$$

Coming back to (3.12), we obtain that

$$-\tilde{r}(T)\big\langle x_n'(T) - x_0'(T), x_n(T) - x_0(T)\big\rangle + \big\langle x_n'(0) - x_0'(0), x_n(0) - x_0(0)\big\rangle \to 0 \\ (n \to \infty)$$

and $x_n' \to x_0'$ in $L^2(I;H)$. Using the boundedness of $|x_n''|$, it follows that $x_n' \to x_0'$ in $C(I;H)$. By $H(\xi)_1$, we have that

$$\omega\bigl|\bigl\|\bigl(x_n(0)-a, x_n(T)-b\bigr)\bigr\|-\bigl\|\bigl(x_0(0)-a, x_0(T)-b\bigr)\bigr\|\bigr|^p \to 0 \quad (n\to\infty). \tag{3.13}$$

Since $\{\|x_n(0)\|\}$ is bounded, passing to a subsequence if necessary, we may assume $x_n(0) \rightharpoonup l$ in H. Then

$$x_n(t) = x_n(0) + \int_0^t x_n'(s)\,ds \rightharpoonup l + \int_0^t x_0'(s)\,ds.$$

Let $v(t) = l + \int_0^t x_0'(s)\,ds$, then $v'(t) = x_0'(t)$, $\forall t \in I$. Because ξ is maximal monotone and $(x_n'(0), -\tilde r(T)x_n'(T)) \in \xi(x_n(0)-a, x_n(T)-b)$, we have

$$\bigl(x_0'(0), -\tilde r(T)x_0'(T)\bigr) \in \xi\bigl(x_0(0)-a, x_0(T)-b\bigr) \cap \xi\bigl(v(0)-a, v(T)-b\bigr).$$

This gives $v(0) = x_0(0)$, $v(T) = x_0(T)$ since ξ is one-to-one. Thus we have

$$\bigl(x_n(0)-a, x_n(T)-b\bigr) \rightharpoonup \bigl(x_0(0)-a, x_0(T)-b\bigr) \quad (n\to\infty).$$

Combining with (3.13) we can have that

$$\bigl(x_n(0)-a, x_n(T)-b\bigr) \to \bigl(x_0(0)-a, x_0(T)-b\bigr)$$

as $H\times H$ is a Hilbert space. Then $x_n(0) \to x_0(0)$ $(n\to\infty)$. Thus, for every $t\in I$, we have

$$\bigl\|x_n(t)-x_0(t)\bigr\| \leqslant \bigl\|x_n(0)-x_0(0)\bigr\| + \int_0^T \bigl\|x_n'(t)-x_0'(t)\bigr\| \to 0 \quad (n\to\infty),$$

which shows $x_n \to x_0$ in $C(I;H)$ and $S(K)$ is relatively compact in $C(I;H)$. Finally, discussing as above, we can also get that the second part of Lemma 3.3 is true. □

Proof of Theorem 3.1. Let $N: C(I;H) \to 2^{L^2(I;H)}$ be the multivalued Nemitsky operator corresponding to F, i.e.,

$$N(x) = \bigl\{f \in L^2(I;H):\ f(t) \in F\bigl(t, x(t)\bigr),\ \text{a.e. } t\in I\bigr\}, \quad \forall x \in C(I;H).$$

Then N has nonempty, weakly compact, convex values in $L^2(I;H)$ and is weakly usc from $C(I;H)$ into $L^2(I;H)$.

To see the nonemptiness, we proceed as follows. Let $x\in C(I;H)$, x_n be step functions with $x_n \to x$ in $C(I;H)$ and w_n be strongly measurable selections of $N(x_n)$. By $H(F)_1$(c), $\{w_n\} \subset L^2(I;H)$ is uniformly integrable. So, passing to a subsequence if necessary, we may assume $w_n \rightharpoonup w$ in $L^2(I;H)$. By Mazur's theorem, there are $\tilde w_n \in \operatorname{conv}\{w_k:\ k\geqslant n\}$ such that $\tilde w_n \to w$ in $L^2(I;H)$, hence $\tilde w_{n_k}(t) \to w(t)$, a.e. on I for some subsequence $\{\tilde w_{n_k}\}_{k\geqslant 1}$. Let $t_0\in I$ be such that $w_n(t_0) \in F(t_0, x_n(t_0))$ for all $n\geqslant 1$ and $\tilde w_{n_k}(t_0) \to w(t_0)$ $(k\to\infty)$. Evidently, such t_0 exists in I almost everywhere. Given $y\in H$ and $\epsilon>0$, then $L(x) = \{\langle y,z\rangle:\ z\in F(t_0,x)\}$ is usc from H into 2^R with compact convex values, and which implies that L is ϵ–δ-usc. Therefore, we obtain that $\langle y, w_n(t_0)\rangle \in L(x_n(t_0)) \subset L(x(t_0)) + (-\epsilon, \epsilon)$, i.e., $\langle y, w_n(t_0)\rangle \in \{\langle y, z\rangle + (-\epsilon, \epsilon)\colon\ z \in F(t_0, x(t_0))\}$ for all large n. As L has compact convex values, we can also get that $\langle y, \tilde{w}_{n_k}(t_0)\rangle \in \{\langle y, z\rangle + (-\epsilon, \epsilon)\colon\ z \in F(t_0, x(t_0))\}$ for all large k. Therefore, in the limit as $k \to \infty$, we have that $\langle y, w(t_0)\rangle \in \{\langle y, z\rangle\colon\ z \in F(t_0, x(t_0))\}$ since $\epsilon > 0$ is arbitrary. Note also that $y \in H$ is arbitrary and F has closed convex values, it is clear that $w(t_0) \in F(t_0, x(t_0))$. Thus we obtain that $N(x) \neq \emptyset$. In fact, the same argument (with $\{u_n\} \subset C(I; H)$ instead of step functions $\{w_n\}$) together with Lemma 2.2 also shows that $N : C(I; H) \to 2^{L^2(I;H)}$ is weakly usc with weakly compact values.

In the following, the proof is given in two steps.

Step 1. Given $\lambda > 0$, and let $N_1(x) = -N(x)$. We consider the following approximation of problem (1.1) at first:

$$\begin{cases} -(p(t)\cdot x_\lambda''(t) + r(t)\cdot x_\lambda'(t)) + \mathcal{A}_\lambda x_\lambda(t) \in N_1(x_\lambda)(t), \quad \text{a.e. } t \in I, \\ (x_\lambda'(0), -\tilde{r}(T)x_\lambda'(T)) \in \xi(x_\lambda(0) - a,\ x_\lambda(T) - b). \end{cases} \tag{3.14}$$

Here $\mathcal{A}$ is the realization of A in E, $\mathcal{A}_\lambda$ is Yosida approximation of $\mathcal{A}$.

Then, for every $f \in L^2(I; H)$, we can define $S_\lambda : L^2(I; H) \to C(I; H)$ by $S_\lambda(f) = x_\lambda$, where x_λ is the solution of problem

$$\begin{cases} -(p(t)\cdot x_\lambda''(t) + r(t)\cdot x_\lambda'(t)) + \mathcal{A}_\lambda x_\lambda(t) = f(t), \quad \text{a.e. } t \in I, \\ (x_\lambda'(0), \tilde{r}(T)x_\lambda'(T)) \in \xi(x_\lambda(0) - a, x_\lambda(T) - b). \end{cases}$$

Obviously, Lemmas 3.2 and 3.3 remain true for operator S_λ.

Now, let $G_\lambda = S_\lambda \circ N_1 : C(I; H) \to 2^{C(I;H)} \setminus \emptyset$, we shall show that there exists a compact convex set $D_\lambda \subset C(I; H)$, such that $G_\lambda : D_\lambda \to 2^{D_\lambda}$.

To this end, take $y \in C(I; H)$ and let $x = G_\lambda(y)$, then x satisfies

$$\begin{cases} -\frac{p}{\tilde{r}}(\tilde{r}x')' + \mathcal{A}_\lambda(x) \in N_1(y), \\ (x'(0), -\tilde{r}(T)x'(T)) \in \xi(x(0) - a, x(T) - b). \end{cases} \tag{3.15}$$

Therefore, as in the proof of Lemma 3.2 (with $u \in N_1(y)$ instead of f, and $\mathcal{A}_\lambda$ instead of $\mathcal{A}$), we can get

$$\big\|x(0)\big\|^p + \|x'\|_2^2 \leqslant C_3 \cdot \big(\|x\|_\infty + \|x\|_\infty \cdot \|u\|_2 + \|u\|_2\big) + C_4, \tag{3.16}$$

where C_3, C_4 is independent of λ as $\|A_\lambda a\| \leqslant \|A^0 a\|$, $\|A_\lambda b\| \leqslant \|A^0 b\|$. According to $H(F)_1$(c), we obtain that $\|u\|_2 \leqslant \|\phi\|_2 + \|y\|_\infty^\alpha \cdot \|\psi\|_2$, so

$$\big\|x(0)\big\|^p + \|x'\|_2^2 \leqslant C_5 \cdot \big(\|x\|_\infty + \|x\|_\infty \cdot \|y\|_\infty^\alpha + \|y\|_\infty^\alpha\big) + C_6, \tag{3.17}$$

here C_5, C_6 is independent of λ, too. Using Cauchy inequality, Young inequality $mn \leqslant \frac{m^p}{p} + \frac{n^q}{q}$ ($m, n > 0$; $p > 1$; $\frac{1}{p} + \frac{1}{q} = 1$), and $\|x\|_\infty \leqslant\ \|x(0)\| + \sqrt{T}\,\|x'\|_2$, we have that

$$\big\|x(0)\big\|^p + \|x'\|_2^2 \leqslant C_7 \cdot \big(\|y\|_\infty^{\alpha q} + \|y\|_\infty^{2\alpha} + \|y\|_\infty^\alpha + 1\big), \tag{3.18}$$

here C_7 is independent of λ and y. So, by $H(\xi)_1$, we can find $M > 0$, which is independent to λ too, such that $\|x\|_\infty \leqslant M$ whenever $\|y\|_\infty \leqslant M$. Let $D_1 = \{x \in C(I; H)\colon\ \|x\|_\infty \leqslant M\}$, then $G_\lambda : D_1 \to 2^{D_1}$, $\forall \lambda > 0$. Next, we shall prove that $G_\lambda(D_1) = \bigcup_{x \in D_1} G_\lambda(x)$ is relatively compact in $C(I; H)$. To this end, let $\{x_n\} \subset G_\lambda(D_1)$. Then there exists $y_n \in D_1$,

$f_n \in N_1(y_n)$, such that $x_n = G_\lambda(y_n) = S_\lambda(f_n)$, $n \geqslant 1$. By $H(F)_1$(c,d), we know that $f_n(t) \in B_M(t)$, $\|f_n(t)\| \leqslant \phi(t) + M\psi(t)$, a.e. $t \in I$. So, applying Lemma 3.3, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ in $C(I; H)$, which shows the relatively compactness of $G_\lambda(D_1)$. Now, let $D_\lambda = \overline{\text{conv}}\, G_\lambda(D_1)$, then $D_\lambda \subset C(I; H)$ is a compact convex set, and satisfies $G_\lambda(D_\lambda) = G_\lambda(\overline{\text{conv}}\, G_\lambda(D_1)) \subseteq G_\lambda(D_1) \subseteq D_\lambda$.

Next, we will show that $G_\lambda : D_\lambda \to 2^{D_\lambda} \setminus \emptyset$ is usc with closed values. Since $D_\lambda \subseteq C(I; H)$ is compact, it suffices to show that $\text{Gr}(G_\lambda)$ is closed. For this purpose, let $x_n = G_\lambda(y_n)$ with $x_n \to x$, $y_n \to y$ in $C(I; H)$ as $n \to \infty$. Then there is $f_n \in N_1(y_n)$, such that $x_n = S_\lambda(f_n)$, $n \geqslant 1$. Because $N_1 : C(I; H) \to 2^{L^2(I;H)}$ is weakly usc with weakly compact convex values, we can choose subsequence $\{f_{n_k}\}$ of $\{f_n\}$ (by Lemma 2.3), such that $f_{n_k} \rightharpoonup f \in N_1(y)$ in $L^2(I; H)$ as $k \to \infty$. Then, applying $H(F)_1$(c,d) and Lemma 3.3, we have that $x_{n_k} \to x_0 = S_\lambda(f)$ in $C(I; H)$, thus $x = x_0 \in G_\lambda(y)$, which implies the closedness of $\text{Gr}(G_\lambda)$.

Therefore, Lemma 2.3 yields a fixed point of G_λ if we are able to prove that G_λ has contractible values. To see this, let $y \in D_\lambda$, and fix $\tilde{x} \in G_\lambda(y)$. Then there exists a unique $\tilde{f} \in N_1(y)$, such that $\tilde{x} = S_\lambda(\tilde{f})$ since $\mathcal{A}_\lambda$ is single valued operator. Let $h : [0, 1] \times G_\lambda(y) \to G_\lambda(y)$ be defined by

$$h(\alpha, x) = S_\lambda\big(\alpha f + (1-\alpha)\tilde{f}\big), \quad \forall (\alpha, x) \in [0, 1] \times G_\lambda(y),$$

here $f \in N_1(y)$ satisfies $x = S_\lambda(f)$. Evidently, such f is also unique and h satisfies

$$h(0, x) = \tilde{x}, \quad h(1, x) = x, \quad \forall x \in G_\lambda(y).$$

Now, we only need to prove that h is continuous. To this end, let $\{(\alpha_n, x_n)\} \subseteq [0, 1] \times G_\lambda(y)$ with $\alpha_n \to \alpha$ in R and $x_n \to x$ in $C(I; H)$. Assume $x_n = S_\lambda(f_n)$, $n \geqslant 1$. Then, according to $H(F)_1$(c,d) and Lemma 3.3, there is $\{f_{n_k}\} \subseteq \{f_n\}$, such that $f_{n_k} \rightharpoonup f_0$ in $L^2(I; H)$ and $x_{n_k} \to x_0 = S_\lambda(f_0)$ in $C(I; H)$ as $k \to \infty$. By the uniqueness of the limit, we can get $x = x_0 = S_\lambda(f_0)$, and f_0 is unique determined by x. Hence $f_n \rightharpoonup f_0$ in $L^2(I; H)$ as $n \to \infty$, which gives $\alpha_n f_n + (1-\alpha_n)\tilde{f} \rightharpoonup \alpha f_0 + (1-\alpha)\tilde{f}$ in $L^2(I; H)$ as $n \to \infty$. Then, using Lemma 3.3 again, we have that $S_\lambda(\alpha_n f_n + (1-\alpha_n)\tilde{f}) \to S_\lambda(\alpha f_0 + (1-\alpha)\tilde{f})$ in $C(I; H)$ as $(\alpha_n, x_n) \to (\alpha, x)$, which gives the continuity of h.

Step 2. Now, for any $\lambda > 0$, we may assume $x_\lambda \in D_\lambda$ is the solution of problem (3.14). Because $M > 0$ is independent of λ, we obtain that $\|x_\lambda\|_\infty \leqslant M$, $\forall \lambda > 0$. Then, according to $H(F)_1$(c), we have that $\|f_\lambda\|_2 \leqslant C_9(\|\phi\|_2 + \|\psi\|_2)$, here $f_\lambda \in N_1(x_\lambda)$, $C_9 > 0$ is independent of λ. As in the proof of Lemma 3.2, we can get that $\|x''_\lambda\|_2$, $\|x'_\lambda\|_2$ and $\|A_\lambda x_\lambda\|_2$ are all bounded. Without loss of generality, we may assume that $f_{\lambda_n} \rightharpoonup f_0$ in $L^2(I; H)$ as $\lambda_n \to 0$. Therefore, using Lemma 3.1, we can get $x_0 \in W^{2,2}(I; H)$, such that

$$\begin{cases} -(p(t) \cdot x''_0(t) + r(t) \cdot x'_0(t)) + \mathcal{A}x_0(t) \ni f_0(t), \quad \text{a.e. } t \in I, \\ (x'_0(0), -\tilde{r}(T)x'_0(T)) \in \xi(x_0(0) - a, x_0(T) - b). \end{cases} \tag{3.19}$$

Multiplying $-\frac{p}{\tilde{r}}(\tilde{r}x'_\lambda - \tilde{r}x'_0)' + \mathcal{A}_\lambda x_\lambda - \mathcal{A}x_0 \ni f_\lambda - f_0$ by $x_\lambda - x_0$ in E, we have that

$$0 \leqslant -\tilde{r}(T)\big\langle x'_\lambda(T) - x'_0(T), x_\lambda(T) - x_0(T)\big\rangle + \big\langle x'_\lambda(0) - x'_0(0), x_\lambda(0) - x_0(0)\big\rangle + \int_0^T \tilde{r}(t)\big\|x'_\lambda(t) - x'_0(t)\big\|^2\,dt$$

$$\leqslant -\int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle A_\lambda x_\lambda(t) - Ax_0(t), \lambda A_\lambda x_\lambda(t)\big\rangle\,dt$$
$$+\int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle f_\lambda(t) - f_0(t), x_\lambda(t) - x_0(t)\big\rangle\,dt.$$

Because $\{f_{\lambda_n}\}_{n\geqslant 1} \subset B_M(t)$, a.e. $t \in I$, $\{x_{\lambda_n}\}_{n\geqslant 1}$ is bounded equicontinuous family in $C(I;H)$, and

$$\left|\int_0^T \frac{\tilde{r}(t)}{p(t)}\big\langle A_{\lambda_n} x_{\lambda_n}(t) - Ax_0(t), \lambda_n A_{\lambda_n} x_{\lambda_n}(t)\big\rangle\,dt\right|$$
$$\leqslant \lambda_n\big(|A_{\lambda_n} x_{\lambda_n}|^2 + C|A_{\lambda_n} x_{\lambda_n}|\big) \to 0 \quad (\lambda_n \to 0),$$

so as in the proof of Lemma 3.3, we can conclude that $x_{\lambda_n} \to x_0$ in $C(I;H)$ as $\lambda_n \to 0$. This in conjunction with $f_{\lambda_n} \rightharpoonup f_0$ in $L^2(I;H)$, $f_{\lambda_n} \in N_1(x_{\lambda_n})$ and Lemma 2.2, yields $f_0 \in N_1(x_0)$, since $N_1 : C(I;H) \to 2^{L^2(I;H)} \setminus \emptyset$ is weakly usc. Thus, by (3.19), we obtain that $x_0 \in W^{2,\,2}(I;H)$ is a solution of problem (1.1). □

We can have a "nonconvex" version of the above existence result. To this purpose, let us put another set of assumptions on the multifunction F.

$H(F)_2$ $F : I \times H \to P_k(H)$ satisfies $H(F)_1$(c,d) and
(a) $(t,x) \to F(t,x)$ is graph measurable;
(b) for a.e. $t \in I$, $x \to F(t,x)$ is lsc.

Theorem 3.2. *Let H is a real separable Hilbert space and $p, r : I \to R$ be continuous functions with $p(t) \geqslant c > 0$, $\forall t \in I$; $a, b \in D(A)$. If $H(A)_1$, $H(F)_2$ and $H(\xi)_1$ hold, then problem* (1.1) *has a solution $x \in W^{2,2}(I;H)$.*

Proof. Let $N : L^2(I;H) \to 2^{L^1(I;H)}$ be the multivalued Nemitsky operator corresponding to F, i.e., $N(x) = \{f \in L^1(I;H)\colon\ f(t) \in F(t,x(t)),\ \text{a.e. } t \in I\}$, $\forall x \in L^2(I;H)$. Then about N we have

Claim 1. There exists a continuous map $g : L^2(I;H) \to L^1(I;H)$, such that $g(x) \in N(x)$, $\forall x \in L^2(I;H)$.

Because of $H(F)_2$, the closedness and decomposability of the values of $N(\cdot)$ is clear. Now we prove that for any $x \in L^2(I;H)$, $N(x) \neq \emptyset$. Let $x \in L^2(I;H)$, then $\phi : I \times H \to I \times H \times H$, $\phi(t,v) = (t, x(t), v)$ is measurable. Because $F(\cdot,\cdot)$ is graph measurable, we have that $\operatorname{Gr} F \in \mathcal{L}(I) \times \mathcal{B}(H) \times \mathcal{B}(H)$. Hence $\phi^{-1}(\operatorname{Gr} F) \in \mathcal{L}(I) \times \mathcal{B}(H)$. Since $\phi^{-1}(\operatorname{Gr} F) = \phi^{-1}\{(t,x,v)\colon\ v \in F(t,x)\} = \operatorname{Gr} F(\cdot, x(\cdot))$, it follows that $F(\cdot, x(\cdot)) : I \to 2^H \setminus \emptyset$ is graph measurable. So we can apply Aumann's selection theorem (see [14, Theorem 5.10]) and obtain $f : I \to H$ a measurable map such that $f(t) \in F(t, x(t))$, a.e. $t \in I$. By virtue of hypothesis $H(F)_1$(c), we obtain $f \in L^1(I;H)$. Therefore $N(x) \neq \emptyset$.

In the following, we will show that $N : L^2(I; H) \to 2^{L^1(I;H)}$ is lsc. To this end, it suffices to show that for every $u \in L^1(I; H)$, $x \mapsto d(u, N(x)) = \inf\{\|u - v\|_1 \colon v \in N(x)\}$ is usc. Because $F(t, x)$ is closed, we have that

$$d\big(u, N(x)\big) = \inf_{v \in N(x)} \int_0^T \big\|u(t) - v(t)\big\|\, dt = \int_0^T \Big(\inf_{v \in F(t, x(t))} \big\|u(t) - v\big\| \Big)\, dt$$
$$= \int_0^T d\big(u(t), F\big(t, x(t)\big)\big)\, dt.$$

$\forall \lambda \geqslant 0$, let $U_\lambda = \{x \in L^2(I; H)\colon d(u, N(x)) \geqslant \lambda\}$. Next, we shall prove that U_λ is closed in $L^2(I; H)$. For this purpose, let $\{x_n\}_{n \geqslant 1} \subset U_\lambda$ with $x_n \to x$ in $L^2(I; H)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $y \in L^2(I; R_+)$, such that $\|x_{n_k}(t)\| \leqslant y(t)$ and $x_{n_k}(t) \to x(t)$, a.e. $t \in I$. By virtue of Fatou's lemma, we have that

$$\lambda \leqslant \limsup d\big(u, N(x_{n_k})\big) = \limsup \int_0^T d\big(u(t), F\big(t, x_{n_k}(t)\big)\big)\, dt$$
$$\leqslant \int_0^T \limsup d\big(u(t), F\big(t, x_{n_k}(t)\big)\big)\, dt \leqslant \int_0^T d\big(u(t), F\big(t, x(t)\big)\big)\, dt = d\big(u, N(x)\big)$$

as $F(t, \cdot)$ is lsc. Then $x \in U_\lambda$, which shows that U_λ is closed in $L^2(I; H)$ and N: $L^2(I; H) \to 2^{L^1(I;H)}$ is lsc. Hence, according to Lemma 2.1 and $L^2(I; H)$ is separable, we can find a continuous map $\tilde{g} : L^2(I; H) \to L^1(I; H)$, such that $\tilde{g}(x) \in N(x)$, $\forall x \in L^2(I; H)$, which proves the claim.

Now, let g be the restriction of $\tilde{g}$ to $C(I; H)$, then $g : C(I; H) \to L^1(I; H)$ is continuous and $g(x) \in N(x) \cap L^2(I; H)$, $\forall x \in C(I; H)$. Let $g_1(x) = -g(x)$. Then it is clear that to finish our proof, we need to solve the following problem:

$$\begin{cases} -(p(t) \cdot x''(t) + r(t) \cdot x'(t)) + \mathcal{A}x(t) \ni g_1(x(t)), & \text{a.e. } t \in I, \\ (x'(0), -\tilde{r}(T)x'(T)) \in \xi(x(0) - a, x(T) - b). \end{cases} \tag{3.20}$$

Because ξ is one-to-one by $H(\xi)_1$, we can define $S : L^2(I; H) \to C(I; H)$ as in Lemma 3.2. Let $G = S \circ g_1$, then $G : C(I; H) \to C(I; H)$ satisfy

Claim 2. $G : C(I; H) \to C(I; H)$ is a compact map.

We first show that G maps bounded sets into relatively compact sets. Let $B \subset C(I; H)$ is a bounded set, then there exists $C > 0$ such that $\|x\|_\infty \leqslant C$, $\forall x \in B$. Then $\|g_1(x)(t)\| \leqslant \|\phi(t)\| + \|\psi(t)\| \cdot C^\alpha$, and $g_1(x)(t) \in k(t) := \{v \in F(t, x)\colon \|x\| \leqslant C\}$, a.e. $t \in I$. By $H(F)_2$(d), we know that $k(t)$ is relatively compact in H. Using Lemma 3.3, we obtain that $S \circ g_1(B)$ is relatively compact in $C(I; H)$.

Next we prove that G is continuous. Let $x_n \to x_0$ in $C(I; H)$, then $\{\|x_n\|_\infty\}$ is bounded and $g_1(x_n) \to g_1(x_0)$ in $L^1(I; H)$. By $H(F)_1$(c), we know that $g_1(x_n)$ is bounded in

$L^2(I;H)$. So, we may assume $g_1(x_n) \rightharpoonup g_1(x_0)$ in $L^2(I;H)$. From Lemma 3.3, we have that $S \circ g_1(x_n) \to S \circ g_1(x_0)$ in $C(I;H)$, which proves the claim.

By the definition of G, it is clear that to finish our proof, we only need to prove that G has a fixed point. As in the proof of Theorem 3.1, we can find a bounded closed convex set $D_1 \subseteq C(I;H)$, such that $G: D_1 \to D_1$. Therefore, we can get a fixed point of G via Schauder's fixed-point theorem and this fixed point is the solution of problem (1.1). □

Remark. It is clear that $p = 2$ satisfies the conditions in $H(\xi)_1$. So, the case that ξ is strongly monotone can be seen as special cases of our work.

4. Applications

In this section, we shall present a characteristic illustration of our results.

Example 4.1. Let Ω be a bounded domain in R^N with smooth boundary Γ, $j: R \to (-\infty, +\infty]$ be an lsc convex proper function with $j(0) = 0$ and $P, r: [0,T] \to R$ be continuous with $P(t) \geqslant c > 0$. We are concerned with the boundary value problem

$$\begin{cases} \big(P(t)\frac{\partial^2 u}{\partial t^2} + r(t)\frac{\partial u}{\partial t}\big)(t,x) \in -\sum_{i=1}^N \frac{\partial}{\partial x_i}\big(\big|\frac{\partial u}{\partial x_i}\big|^{p-2}\frac{\partial u}{\partial x_i}\big) + F(t, u(t,\cdot)), \\ \quad \text{a.e. in } [0,T]\times\Omega, \\ -\sum_{i=1}^N \big|\frac{\partial u}{\partial x_i}\big|^{p-2}\frac{\partial u}{\partial x_i}(t,x)\cos(\vec{n},\vec{e}_i) \in \partial j(u(t,x)) \\ \quad \text{a.e. on } [0,T]\times\Gamma, \\ \big(\frac{\partial u}{\partial t}(0,x), -\tilde{r}(T)\frac{\partial u}{\partial t}(T,x)\big) \in (\partial g + \omega I)(u(0,x)-a(x), u(T,x)-b(x)), \\ \quad \text{a.e. on } \Omega, \end{cases} \tag{4.1}$$

where $a, b \in L^2(\Omega)$, $\omega > 0$, $p \geqslant 2$, $\tilde{r}(t) = \exp(\int_0^t r(s)/p(s)\,ds)$, $F: [0,T] \times L^2(\Omega) \to 2^{L^2(\Omega)} \setminus \emptyset$ is a multifunction, $g: L^2(\Omega) \times L^2(\Omega) \to (-\infty, \infty]$ is an lsc convex proper functional with $g(0,0) = 0$, $\vec{n} = \vec{n}(x)$ is the outward normal derivative to Γ at $x \in \Gamma$ and $\{\vec{e}_i\colon i = 1, \ldots, N\}$ is the canonical base of R^N.

For problem (4.1), let $A: D(A) \subset L^2(\Omega) \to L^2(\Omega)$ be defined by

$$Au = -\Delta_p u = -\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right), \quad p \geqslant 2,$$

with

$$D(A) = \left\{u \in W^{1,p}(\Omega)\colon\ \Delta_p u \in L^2(\Omega),\ -\frac{\partial u}{\partial \vec{n}_p} \in \partial j\big(u(x)\big),\ \text{a.e. on } \Gamma\right\},$$

where

$$\frac{\partial u}{\partial \vec{n}_p} = \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\cos(\vec{n},\vec{e}_i).$$

It is well known that A is maximal monotone.

Now, using Theorem 3.2, we can have the following corollary.

Corollary 4.1. *Let A, $D(A)$ and ∂g be defined as above. If A is ∂g-monotone and $F: I \times L^2(\Omega) \to 2^{L^2(\Omega)} \setminus \emptyset$ satisfies $H(F)_2$, then for every $a, b \in D(A)$, $\omega > 0$, the boundary value problem* (4.1) *has a solution $u \in W^{2,2}([0,T]; L^2(\Omega))$.*

Remarks. (1) For $p = 2$, A given as above is just the Laplace operator in R^N with

$$D(\Delta) = \left\{ u \in H^2(\Omega)\colon -\frac{\partial u}{\partial \vec{n}_2} \in \partial j\big(u(x)\big), \text{ a.e. on } \Gamma \right\}.$$

(2) The map satisfying $H(F)_1$ (respectively $H(F)_2$) exists. For example, let $F: I \times L^2(\Omega) \to L^2(\Omega)$ be defined by

$$F\big(t, u(x)\big) = \left| \int_\Omega K(t,x,y)u(y)\,dy \right|^\alpha, \quad \text{a.e. } (t,x) \in I \times \Omega,\ 0 < \alpha < 1,$$

here $K \in L^2(0,T; L^2(\Omega \times \Omega))$, then we can show that F satisfies $H(F)_2$. In fact, for every $u \in L^2(\Omega)$, we have that

$$\begin{aligned}
\big\| F\big(t, u(\cdot)\big) \big\|_2^2 &= \int_\Omega \left| \int_\Omega K(t,x,y)u(y)\,dy \right|^{2\alpha} dx \\
&\leqslant \int_\Omega \left(\int_\Omega \big|K(t,x,y)\big|^2\,dy \right)^\alpha \left(\int_\Omega \big|u(y)\big|^2\,dy \right)^\alpha dx \\
&\leqslant (\operatorname{mes}\Omega)^{1-\alpha} \cdot \left(\int_\Omega \int_\Omega \big|K(t,x,y)\big|^2\,dy\,dx \right)^\alpha \cdot \|u\|_2^{2\alpha},
\end{aligned}$$

which implies that for a.e. $t \in I$, $F(t,\cdot): L^2(\Omega) \to L^2(\Omega)$ is continuous and satisfies $H(F)_1$(a–c). Furthermore, if $u_n \rightharpoonup u$ in $L^2(\Omega)$, then for a.e. $t \in I$, we have that

$$\begin{aligned}
&\big\| F(t,u_n) - F(t,u) \big\|_2^2 \\
&\quad = \int_\Omega \left(\left| \int_\Omega K(t,x,y)u_n(y)\,dy \right|^\alpha - \left| \int_\Omega K(t,x,y)u(y)\,dy \right|^\alpha \right)^2 dx,
\end{aligned}$$

and

$$\left| \int_\Omega K(t,x,y)u_n(y)\,dy \right|^{2\alpha} \leqslant \left(\int_\Omega \big|K(t,x,y)\big|^2\,dy \right)^\alpha \cdot \|u_n\|_2^{2\alpha},$$

$$\int_\Omega K(t,x,y)u_n(y)\,dy \to \int_\Omega K(t,x,y)u(y)\,dy \quad (n \to \infty), \text{ a.e. } x \in \Omega.$$

So, according to Lebesgue's control convergence theorem, we have that $F(t,u_n) \to F(t,u)$, a.e. $t \in I$ in $L^2(\Omega)$. Thus F satisfies $H(F)_1$(d).

(3) Let $g : L^2(\Omega) \times L^2(\Omega) \to R$ be defined by $g(u,v) = \|u\|_2 + \|v\|_2$, then $g(0,0) = 0$. Simple calculus shows that, if $J_\lambda = (I + \lambda A)^{-1}$, then

$$g(J_\lambda x - J_\lambda y, J_\lambda w - J_\lambda z) \leqslant g(x - y, w - z), \quad \forall \lambda > 0,\ x, y, w, z \in L^2(\Omega).$$

Therefore, we obtain that g satisfies the conditions of Corollary 4.1.

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